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On the Perron roots of principal submatrices of co-order one of irreducible nonnegative matrices ☆

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Abstract

Let A be an irreducible nonnegative matrix and $\lambda(A)$ be the Perron root (spectral radius) of A. Denote by $\lambda_{\min}(A)$ the minimum of the Perron roots of all the principal submatrices of co-order one. It is well known that the interval $(\lambda_{\min}(A), \lambda(A))$ does not contain any eigenvalues of A. Consider any principal submatrix $A - v$ of co-order one whose Perron root is equal to $\lambda_{\min}(A)$. We show that the Jordan structure of $\lambda_{\min}(A)$ as an eigenvalue of A is obtained from that of the Perron root of $A - v$ as follows: one largest Jordan block disappears and the others remain the same. So, if only one Jordan block corresponds to the Perron root of the submatrix, then $\lambda_{\min}(A)$ is not an eigenvalue of A. By Schneider's theorem, this holds if and only if there is a Hamiltonian chain in the singular digraph of $A - v$. In the general case the Jordan structure for the Perron root of the submatrix $A - v$ and therefore that for the eigenvalue $\lambda_{\min}(A)$ of A can be arbitrary. But if the Perron root $\lambda(A - w)$ of a principal submatrix $A - w$ of co-order one is strictly greater than $\lambda_{\min}(A)$, then $\lambda(A - w)$ is a simple eigenvalue of $A - w$. We also obtain different representations for the generalized eigenvectors corresponding to the eigenvalues of A contained in the annulus $\{\lambda\colon \lambda_{\min}(A) < |\lambda| < \lambda(A)\}$.

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1. Introduction

This paper is devoted to the relationships between the spectral properties of an irreducible matrix A with nonnegative entries and those of its principal submatrices of co-order one. Any such submatrix has the form $A - v$, where the sign "minus" means deleting the vth row and vth column of A. It is well known [1] (see also [2]) that any nonnegative matrix A has a nonnegative eigenvalue $\lambda(A)$, the so-called Perron root, which is equal to its spectral radius. From Frobenius' proof [3] (see also [4]) of the Perron theorem it follows that for any v the interval $(\lambda(A - v), \lambda(A))$ does not contain points of the spectrum of the matrix A. Of course, the same is true for the interval $(\lambda_{\min}(A), \lambda(A))$, where $\lambda_{\min}(A)$ is the minimum of the Perron roots of all the principal submatrices of co-order one. What can one say about the spectral properties of the point $\lambda_{\min}(A)$ as an eigenvalue of the original irreducible matrix A? When is this point the second-biggest real eigenvalue of A? In our paper we completely reduce this problem to the determination of the spectral properties of the Perron root for a principal submatrix $A - v$ of co-order one whose spectral radius is equal to $\lambda_{\min}(A)$. This can be useful because in many cases the Jordan form for the Perron root of a nonnegative matrix is simply determined by its singular digraph (see [5]). In particular, Theorem 2 of our paper and Schneider's theorem 6.5 [5] allow us to answer the second question above, which goes back to Frobenius. The answer (Corollary 3) is very simply formulated: the Perron root of $A - v$ belongs to the spectrum of the matrix A if and only if there is no path going through all the vertices of the singular digraph of the submatrix $A - v$.

Our paper is organized as follows. Section 2 is devoted to an arbitrary matrix with complex entries. We show how the sizes of the Jordan blocks of a fixed eigenvalue are changed under the transition from a principal submatrix of co-order one to the original matrix when the algebraic multiplicitity decreases by the size of the largest Jordan block. It turns out that in this case one largest Jordan block disappears and all the others remain the same (Theorem 1). Moreover, we explain why such a change of the spectral properties is typical. In Section 3 it is proved that this typical change takes place for the Perron root of any principal submatrix of co-order one of an irreducible matrix with nonnegative entries (Theorem 2). Section 4 is devoted to the spectral properties of the Perron roots of principal submatrices of co-order one themselves. We show that if the Perron root of $A - v$ is strictly greater than $\lambda_{\min}(A)$, then $A - v$ inherits some spectral properties of an irreducible matrix: the Perron root $\lambda(A - v)$ is a simple eigenvalue of $A - v$ (Proposition 1). On the other hand, there are no restrictions to the sizes of the Jordan blocks of a principal submatrix of co-order one whose Perron root is equal to $\lambda_{\min}(A)$. But the spectral properties of the Perron roots of any two such principal submatrices of the same irreducible matrix are consistent with each other. In fact, they are almost the same (Proposition 2).

In our paper we are concerned not only about the spectral properties of the point $\lambda_{\min}(A)$, but also about those of the eigenvalues contained in the annulus $\{\lambda\colon \lambda_{\min}(A) < |\lambda| \leqslant \lambda(A)\}$. In the Appendix we study the spectral properties of an eigenvalue

λ of an arbitrary matrix A that does not belong to the spectrum of some submatrix $A - v$. It is easy to show that any such eigenvalue λ has geometric multiplicity one. Numerous equivalent expressions for the associated eigenvector are well known and often used (e.g., see [6,7]). In our paper we construct the whole Jordan chain corresponding to such an eigenvalue. As a consequence of these results one can obtain an expression for the entries of the generalized eigenvectors corresponding to the eigenvalues contained in the annulus $\{\lambda\colon \lambda_{\min}(A) < |\lambda| \leqslant \lambda(A)\}$ in terms of the generating functions of the first return times and their derivatives. Such a representation has been known only for positive Perron eigenvectors (see [8]) and extensively used in the theory of Markov chains. We hope that our new results will be also useful for studying the convergence of the transition probabilities to their ergodic limits.

2. The typical change of the spectral properties of a fixed eigenvalue

The spectral properties of an eigenvalue λ of an arbitrary matrix M are characterized by the sequence of the sizes of its Jordan blocks $n_1 \leqslant \cdots \leqslant n_k$. The *index* of λ, $\operatorname{ind}_M(\lambda) = n_k$ (i.e., the size of the largest Jordan block of λ), the *algebraic multiplicity* $n_M(\lambda) = n_1 + \cdots + n_k$ and the *geometric multiplicity* $g_M(\lambda) = k$ play an essential role in the spectral matrix theory. So we give explicit definitions of them here. A vector ξ is called *a generalized eigenvector* of M corresponding to the eigenvalue λ if there is a number m such that $(M - \lambda E)^m \xi = 0$, where E is the identity matrix of appropriate order. Denote by $L_M(\lambda)$ the space of all such vectors. It is called *the generalized eigenspace* of the matrix M corresponding to the eigenvalue λ. The number m such that $(M - \lambda E)^m \xi = 0$ and $(M - \lambda E)^{m-1} \xi \neq 0$ is the *height* of the generalized eigenvector ξ. If $m = 1$, then ξ is called *an eigenvector* of M associated with λ. All such vectors form a subspace of the space $L_M(\lambda)$. This subspace is called *the eigenspace* of M corresponding to λ. Its dimension is equal to $g_M(\lambda)$. Moreover, after introducing the above definitions, we have that $n_M(\lambda) = \dim L_M(\lambda)$ and $\operatorname{ind}_M(\lambda)$ is the smallest number q such that $(M - \lambda E)^q \xi = 0$ for any $\xi \in L_M(\lambda)$. We remark that any of the conditions $n_M(\lambda) = 0$, $\operatorname{ind}_M(\lambda) = 0$, and $g_M(\lambda) = 0$ means that λ is not an eigenvalue of M.

Let A be an arbitrary square matrix and $V(A)$ be the set of its indices. For any $v \in V(A)$, denote by $A - v$ the matrix obtained from A by deleting the vth column and vth row. It is clear that the order of $A - v$ is one less than that of A. We shall say that $A - v$ is a principal submatrix of *co-order one*. Let η_v and ξ_v be the vectors obtained from the vth column and vth row of A, respectively, by deleting the diagonal entry $A(v, v)$. In the proofs of our results we shall always assume that v is the first index of A. In this case the matrix A has the following form:

$$\begin{pmatrix} A(v, v) & \xi_v \\ \eta_v & A - v \end{pmatrix}.$$

This partition will be very useful for us because of we shall construct Jordan chains of A from those of $A - v$.

The action of any matrix A on a vector ζ can be described by means of the scalar product $(\cdot,\cdot)$ which is defined for any two vectors ξ and η in the following way: $(\xi,\eta) = \sum_w \xi(w)\eta(w)$. Indeed, in this case the vth entry of the vector $A\zeta$ is the scalar product of the vth row of A and the vector ζ. So we shall use the above bilinear form instead of the usual scalar product: $(\xi,\eta)_{\mathbb{C}} = \sum_w \xi(w)\overline{\eta(w)}$.

The following identity was used by Frobenius [3] in his own proof of the Perron theorem [1] (see also [2,4]). The present form of the identity can be found in [9] (see Lemma 1 therein).

Lemma 1. *Let A be an arbitrary matrix. Then for any $v \in V(A)$ we have*

$$\det(zE - A) = (z - A(v,v) - (\xi_v, (zE - (A - v))^{-1}\eta_v)) \times \det(zE - (A - v)). \tag{1}$$

Remark 1. The first factor of the product in the right side of (1) is called *the Schur complement* of $zE - (A - v)$ in $zE - A$.

The following result is a simple consequence of Lemma 1, the Laurent expansion of the resolvent, and the definition of the index of an eigenvalue.

Corollary 1. *Let A be an arbitrary matrix and v be any index of A. Then for any eigenvalue λ of the submatrix $A - v$ we have*

$$n_A(\lambda) \geqslant n_{A-v}(\lambda) - \mathrm{ind}_{A-v}(\lambda). \tag{2}$$

Proof. Let $p_{A,v}(\lambda)$ be the order of the pole of the first factor in (1) at the point $z = \lambda$ (if this point is a zero of order m of the function, then we assume that $p_{A,v}(\lambda) = -m$). Then the equality $n_A(\lambda) = n_{A-v}(\lambda) - p_{A,v}(\lambda)$ holds. So we must only show that $p_{A,v}(\lambda) \leqslant \mathrm{ind}_{A-v}(\lambda)$.

Set $q = \mathrm{ind}_{A-v}(\lambda)$. The resolvent of the matrix $A - v$ admits the Laurent expansion in a sufficiently small neighborhood of the point $z = \lambda$ (see [10], Chapter 1, Section 3; our form is close to that of [11]):

$$(zE - (A - v))^{-1} = \sum_{m=1}^{q} Q_{-m}(z - \lambda)^{-m} + R(z). \tag{3}$$

Here Q_{-1} is the projection operator on the generalized eigenspace $L_{A-v}(\lambda)$ along the direct sum of all the generalized eigenspaces of the matrix $A - v$ corresponding to its eigenvalues other than λ, $Q_{-m} = ((A - v) - \lambda E)^{m-1}Q_{-1}$, and $R(z)$ is an analytic operator in some neighborhood of the point $z = \lambda$. Using (3), we obtain that

$$(\xi_v, (zE - (A - v))^{-1}\eta_v) = \sum_{m=1}^{q} (\xi_v, Q_{-m}\eta_v)(z - \lambda)^{-m} + R_*(z), \tag{4}$$

where $R_*(z)$ is an analytic function in some neighborhood of the point $z = \lambda$. It is clear now that the order of the pole of $(\xi_v, (zE - (A - v))^{-1}\eta_v)$ at $z = \lambda$ is not greater than q. The corollary is proved. $\square$

It is evident that the equality $n_A(\lambda) = n_{A-v}(\lambda) - \mathrm{ind}_{A-v}(\lambda)$ takes place if and only if $(\xi_v, Q_{-q}\eta_v) \neq 0$. The equation $(\xi_v, Q_{-q}\eta_v) = 0$ determines some surface of second order in the space whose coordinates are the entries of vth row and vth column (here we consider the submatrix $A - v$ and therefore the matrix corresponding to the operator Q_{-q} as fixed matrices). This surface has Lebesgue measure zero. In other words, the term $(\xi_v, Q_{-q}\eta_v)$ is not equal to zero for almost all vectors ξ_v and η_v (here we consider ξ_v and η_v as vectors whose entries are random variables). So, the situation when the algebraic multiplicity of a fixed eigenvalue λ decreases by the index of λ under passing from $A - v$ to A is *typical* in both algebraic and probability senses.

Theorem 1. *Let A be an arbitrary matrix and v be any index of A. Let λ be any eigenvalue of the submatrix $A - v$ and $n_1 \leqslant \cdots \leqslant n_k$ be the sizes of its Jordan blocks. Assume that*

$$n_A(\lambda) = n_{A-v}(\lambda) - \mathrm{ind}_{A-v}(\lambda).$$

Then $n_1, \ldots, n_{k-1}$ are the sizes of the Jordan blocks for λ as an eigenvalue of the original matrix A.

Proof. In the case of $k = 1$ the statement of the theorem is evident. Suppose that $k \geqslant 2$. First, for any $p = 1, \ldots, k-1$, we shall construct a chain $\xi_1^{(p)*}, \ldots, \xi_{n_p}^{(p)*}$ such that $(A - \lambda E)\xi_n^{(p)*} = \xi_{n-1}^{(p)*}$ for $n = 1, \ldots, n_p$, where $\xi_0^{(p)*} = 0$. Let $\xi_n^{(p)}$, $n = 1, \ldots, n_p$, $p = 1, \ldots, k$, be any Jordan basis of the generalized eigenspace $L_{A-v}(\lambda)$ (here and everywhere in the sequel we assume that $((A - v) - \lambda E)\xi_n^{(p)} = \xi_{n-1}^{(p)}$ for $n = 1, \ldots, n_p$ and $\xi_0^{(p)} = 0$). Consider the condition $(\xi_v, Q_{-q}\eta_v) \neq 0$ in detail.

Assume that $n_1 \leqslant \cdots \leqslant n_d = n_{d+1} = \cdots = n_k$. So, $k - d + 1$ Jordan blocks have the largest size $q = n_d = \cdots = n_k$. Choose the eigenvectors $\eta_1^{(d)}, \ldots, \eta_1^{(k)}$ belonging to the longest Jordan chains of the transpose-matrix $(A - v)^{\mathrm{T}}$ associated with λ such that $(\eta_1^{(p)}, \xi_{n_{p'}}^{(p')}) = \delta_{pp'}$, where $p, p' = d, \ldots, k$ and $\delta_{pp'}$ is Kronecker symbol. In this case for any $n' \leqslant q - 1$ and $p', p = d, \ldots, k$ we have

$$\begin{aligned}\left(\eta_1^{(p)}, \xi_{n'}^{(p')}\right) &= \left(\eta_1^{(p)}, ((A - v) - \lambda E)\xi_{n'+1}^{(p')}\right) \\ &= \left(\left((A - v)^{\mathrm{T}} - \lambda E\right)\eta_1^{(p)}, \xi_{n'+1}^{(p')}\right) = 0,\end{aligned}$$

because $((A - v)^{\mathrm{T}} - \lambda E)\eta_1^{(p)} = 0$ for $p = d, \ldots, k$. Let $\eta_q^{(d)}, \ldots, \eta_q^{(k)}$ be any vectors such that $((A - v)^{\mathrm{T}} - \lambda E)^{q-1}\eta_q^{(p)} = \eta_1^{(p)}$ for $p = d, \ldots, k$. Then for $p = d, \ldots, k$ and $p' = 1, \ldots, d - 1$ we have

$$\left(\eta_1^{(p)}, \xi_{n'}^{(p')}\right) = \left(\left((A-v)^{\mathrm{T}} - \lambda E\right)^{q-1} \eta_q^{(p)}, \xi_{n'}^{(p')}\right)$$
$$= \left(\eta_q^{(p)}, \left((A-v) - \lambda E\right)^{q-1} \xi_{n'}^{(p')}\right) = 0,$$

because $((A-v)-\lambda E)^{q-1}\xi_{n'}^{(p')} = 0$ for such choice of p'. Moreover, since the restriction of $(A-v)-\lambda E$ to the generalized eigenspace $L_{A-v}(\mu)$, where $\mu \neq \lambda$, is invertible, for any $\xi \in L_{A-v}(\mu)$ there is a vector $\xi' \in L_{A-v}(\mu)$ such that $\xi = ((A-v)-\lambda E)\xi'$. This fact implies the equality

$$\left(\eta_1^{(p)}, \xi\right) = \left(\eta_1^{(p)}, ((A-v)-\lambda E)\xi'\right) = \left(((A-v)^{\mathrm{T}} - \lambda E)\eta_1^{(p)}, \xi'\right) = 0.$$

So we can write $Q_{-1} = \sum_{p=d}^{k} (\eta_1^{(p)}, \cdot)\xi_{n_p}^{(p)} + \widetilde{Q}_{-1}$, where $((A-v)-\lambda E)^{q-1}\widetilde{Q}_{-1} = 0$.

Then

$$Q_{-q} = ((A-v)-\lambda E)^{q-1} Q_{-1} = \sum_{p=d}^{k} (\eta_1^{(p)}, \cdot)\xi_1^{(p)}$$

and therefore the condition $(\xi_v, Q_{-q}\eta_v) \neq 0$ means that $\sum_{p=d}^{k}\left(\xi_v, \xi_1^{(p)}\right)\left(\eta_v, \eta_1^{(p)}\right) \neq 0$. In particular, $(\xi_v, \xi_1^{(p)}) \neq 0$ for some $p = d, \ldots, k$. Without loss of generality, one can assume that the inequality $(\xi_v, \xi_1^{(k)}) \neq 0$ holds. In this case for any ξ there is a unique number α satisfying the equation $(\xi_v, \xi - \alpha\xi_1^{(k)}) = 0$.

Let $\alpha_1^{(p)}, \ldots, \alpha_n^{(p)}, \ldots, \alpha_{n_p}^{(p)}$ be the sequence such that $\alpha_1^{(p)} = (\xi_v, \xi_1^{(p)})/(\xi_v, \xi_1^{(k)})$ and for any $n = 2, \ldots, n_p$ the number $\alpha_n^{(p)}$ is the unique solution to the equation

$$\left(\xi_v, \xi_n^{(p)} - \sum_{h=1}^{n-1} \alpha_h^{(p)} \xi_{n+1-h}^{(k)} - \alpha_n^{(p)}\xi_1^{(k)}\right) = 0.$$

Then for any $n = 1, \ldots, n_p$ the vector

$$\xi_n^{(p)\prime} = \xi_n^{(p)} - \sum_{h=1}^{n} \alpha_h^{(p)} \xi_{n+1-h}^{(k)}$$

satisfies the condition $(\xi_v, \xi_n^{(p)\prime}) = 0$. Moreover, for any $n = 1, \ldots, n_p$ we have

$$\left((A-v)-\lambda E\right)\xi_n^{(p)\prime} = \left((A-v)-\lambda E\right)\left\{\xi_n^{(p)} - \sum_{h=1}^{n} \alpha_h^{(p)} \xi_{n+1-h}^{(k)}\right\}$$
$$= \xi_{n-1}^{(p)} - \sum_{h=1}^{n-1} \alpha_h^{(p)} \xi_{n-h}^{(k)} = \xi_{n-1}^{(p)\prime}.$$

Denote by $\xi_n^{(p)*}$ the vector whose vth entry is equal to zero and whose restriction to the set $V(A-v)$ coincides with $\xi_n^{(p)\prime}$.

We see that the vector $\xi_n^{(p)*}$ satisfies the equation $(A - \lambda E)\xi_n^{(p)*} = \xi_{n-1}^{(p)*}$:

$$\begin{pmatrix} A(v,v) & \xi_v \\ \eta_v & A-v \end{pmatrix} \begin{pmatrix} 0 \\ \xi_n^{(p)\prime} \end{pmatrix} = \begin{pmatrix} \left(\xi_v, \xi_n^{(p)\prime}\right) \\ (A-v)\xi_n^{(p)\prime} \end{pmatrix} = \lambda \begin{pmatrix} 0 \\ \xi_n^{(p)\prime} \end{pmatrix} + \begin{pmatrix} 0 \\ \xi_{n-1}^{(p)\prime} \end{pmatrix}$$

and therefore $\xi_1^{(p)*}, \ldots, \xi_{n_p}^{(p)*}$ is the required chain. In all we obtain $k-1$ chains $\xi_1^{(p)*}, \ldots, \xi_{n_p}^{(p)*}$, where $p = 1, \ldots, k-1$. Their lengths are $n_1, \ldots, n_{k-1}$. It is clear that they all consist only of linearly independent generalized eigenvectors of A corresponding to λ. But we cannot state yet that each of them is a Jordan chain of the original matrix A.

By the condition $n_A(\lambda) = n_{A-v}(\lambda) - \mathrm{ind}_{A-v}(\lambda)$, the algebraic multiplicity of the eigenvalue λ of the matrix A is equal to $n_1 + \cdots + n_{k-1}$. Hence the number of the vectors belonging to the chains constructed above is equal to the dimension of the generalized eigenspace of the matrix A corresponding to the eigenvalue λ. From this it follows that all the vectors $\xi_1^{(p)*}, \ldots, \xi_{n_p}^{(p)*}$, where $p = 1, \ldots, k-1$, form a basis and therefore a Jordan basis of the generalized eigenspace $L_A(\lambda)$. The theorem is proved. □

Remark 2. By the proof of Theorem 1, if $n_A(\lambda) = n_{A-v}(\lambda) - \mathrm{ind}_{A-v}(\lambda)$, then the vth entry of any generalized eigenvector of A corresponding to λ is equal to zero. So, in the case when λ is an eigenvalue of A, the typical change of the spectral properties cannot take place under the transition from each principal submatrix of co-order one to the original matrix A.

3. The spectral properties of the Perron root of a principal submatrix of co-order one as an eigenvalue of the original matrix

In the remaining part of our paper we shall consider matrices with nonnegative entries. Any such matrix M has a nonnegative eigenvalue, the so-called Perron root $\lambda(M)$, which is equal to its spectral radius. When M is also irreducible, Frobenius showed in [12] (see also [2]) that this eigenvalue is simple and the corresponding eigenvector can be chosen in a such way that all its entries will be positive. Let $G(M)$ be the weighted digraph of the matrix M. By definition, the set of its vertices coincides with the set $V(M)$ of indices of M and (w, w') is an arc of $G(M)$ iff $M(w, w') > 0$. In this case the weight of the arc (w, w') is equal to $M(w, w')$. A strong component of the weighted digraph $G(M)$ is called the *Perron* strong component if the Perron root of the principal submatrix corresponding to the strong component is equal to $\lambda(M)$. It is well known that the algebraic multiplicity of the Perron root of M is equal to the number of the Perron strong components of $G(M)$ (see the second statement of Theorem 8.1 [5]). The generalized eigenspace corresponding to the Perron root is called *the Perron generalized eigenspace*.

Let $\gamma = \{w_i\}_{i=0}^m$ be a walk in the digraph $G(M)$. Denote by $\ell(\gamma)$ its length (which is equal to m) and set

$$Q_M(\gamma) = M(w_0, w_1)M(w_1, w_2) \cdots M(w_{m-1}, w_m).$$

So, the sequence $\gamma = \{w_i\}_{i=0}^m$ is a walk in $G(M)$ iff $Q_M(\gamma) > 0$. Assume that $\gamma_1 = \{w_i\}_{i=0}^m$ and $\gamma_2 = \{w_i'\}_{i=0}^p$ are two walks in $G(M)$ for which $w_m = w_0'$. In this case we can define their product $\gamma_1\gamma_2$ as the walk $\{w_i''\}_{i=0}^{m+p}$ such that $w_i'' = w_i$ for $i = 0, \ldots, m$ and $w_{m+i}'' = w_i'$ for $i = 0, \ldots, p$. It is clear that $\ell(\gamma_1\gamma_2) = \ell(\gamma_1) + \ell(\gamma_2)$ and $Q_M(\gamma_1\gamma_2) = Q_M(\gamma_1)Q_M(\gamma_2)$.

In this section we shall use the following lemma, which is really a simple consequence of the famous Friedland–Schneider theorem on the growth of the entries of powers of a nonnegative matrix [13].

Lemma 2. *Let M be an arbitrary nonnegative matrix, and w and w' be any two indices of M. Assume that w_* and w_*' are indices of the matrix M for which there exist paths γ_{w_*w} and $\gamma_{w'w_*'}$ in $G(M)$ from w_* to w and from w' to w_*', respectively. Then the order of the pole of the function $(zE - M)^{-1}(w_*, w_*')$ at the point $z = \lambda(M)$ is not less than that of the function $(zE - M)^{-1}(w, w')$.*

Proof. It is well known that

$$(zE - M)^{-1} = \sum_{n=0}^{\infty} M^n/z^{n+1}$$

for $|z| > \lambda(M)$. Thus, for any w and w' we have

$$(zE - M)^{-1}(w, w') = \sum_{n=0}^{\infty} M^n(w, w')/z^{n+1}$$

in the domain $|z| > \lambda(M)$. So in our case it is sufficient to show that

$$M^{n+p}(w_*, w_*') \geqslant cM^n(w, w')$$

for some natural p and positive number c.

The (w, w')-entry $M^n(w, w')$ of the matrix M^n is equal to the sum of $Q_M(\gamma)$ over all the walks γ of length n that pass from w to w' in the digraph $G(M)$. If γ is a walk in $G(M)$ from w to w', then $\gamma_{w_*w}\gamma\gamma_{w'w_*'}$ is a walk in $G(M)$ from w_* to w_*'. Thus,

$$M^{n+\ell(\gamma_{w_*w})+\ell(\gamma_{w'w_*'})}(w_*, w_*') \geqslant Q_M(\gamma_{w_*w})M^n(w, w')Q_M(\gamma_{w'w_*'}).$$

So, we have the desired inequality with

$$p = \ell(\gamma_{w_*w}) + \ell(\gamma_{w'w_*'}) \quad \text{and} \quad c = Q_M(\gamma_{w_*w})Q_M(\gamma_{w'w_*'}).$$

The lemma is proved. $\square$

Let A be an irreducible nonnegative matrix. Then $\lambda(A-v)<\lambda(A)$ for any $v \in V(A)$ (see [2]; the converse statement is also true: if all the entries of A are nonnegative and $\lambda(A-v)<\lambda(A)$ for any $v \in V(A)$, then the matrix A must be irreducible). This means that the point $\lambda(A)$ does not belong to the spectrum of the matrix $A-v$. So, for the Perron root $\lambda(A)$ of the matrix A we have the following equation:

$$\lambda(A)-A(v,v)-(\xi_v,(\lambda(A)E-(A-v))^{-1}\eta_v)=0.$$

Since all the entries of the vectors ξ_v and η_v are nonnegative and any matrix entry of $(zE-(A-v))^{-1}$ is a nonincreasing nonnegative function on the interval $(\lambda(A-v),\infty)$ (see [2]), the inequality

$$z-A(v,v)-(\xi_v,(zE-(A-v))^{-1}\eta_v)<0$$

holds for any $z \in (\lambda(A-v),\lambda(A))$. From this and identity (1) it follows that for any $v \in V(A)$ the interval $(\lambda(A-v),\lambda(A))$ does not contain any eigenvalues of the original matrix A (for a matrix with positive entries this fact was known to Frobenius [3]). The following result shows how the spectral properties of the point $\lambda(A-v)$ as an eigenvalue of the original matrix A depend on those of $\lambda(A-v)$ as an eigenvalue of the submatrix $A-v$ itself.

Theorem 2. *Let A be an irreducible nonnegative matrix, v be any index of A, and $n_1 \leqslant \cdots \leqslant n_k$ be the sizes of the Jordan blocks of $A-v$ for its Perron root $\lambda(A-v)$. Then $n_1 \leqslant \cdots \leqslant n_{k-1}$ are the sizes of the Jordan blocks corresponding to the point $\lambda(A-v)$ as an eigenvalue of the original matrix A.*

Proof. Set $q=\operatorname{ind}_{A-v}(\lambda(A-v))$. By Theorem 1, it is sufficient to show that the order of the pole of the function $(\xi_v,(zE-(A-v))^{-1}\eta_v)$ at the point $z=\lambda(A-v)$ is equal to q. Rewrite $(\xi_v,(zE-(A-v))^{-1}\eta_v)$ as

$$\sum_{w\neq v}\sum_{w'\neq v}A(v,w)(zE-(A-v))^{-1}(w,w')A(w',v)$$

and consider the Laurent expansion

$$(zE-(A-v))^{-1}(w,w')=Q_{-q}(w,w')(z-\lambda(A-v))^{-q}+\cdots$$

for any entry $(zE-(A-v))^{-1}(w,w')$ of the resolvent. Since the matrix $(zE-(A-v))^{-1}$ is nonnegative for any $z>\lambda(A-v)$, we have $Q_{-q}(w,w')\geqslant 0$ (we remark that for at least one pair of indices this coefficient must be positive). Thus, in our case it is sufficient to show that there are indices w_* and w'_* for which $A(v,w_*)Q_{-q}(w_*,w'_*)A(w'_*,v)>0$ (the existence of such indices is evident in the case of a matrix with positive entries).

Let w and w' be any indices such that the function $(zE-(A-v))^{-1}(w,w')$ has a pole of order q at the point $z=\lambda(A-v)$. Since the matrix A is irreducible, there are simple paths $\gamma=\{w_i\}_{i=0}^{m}$ and $\gamma'=\{w'_{i'}\}_{i'=0}^{p}$ in the digraph $G(A)$

such that $w_0 = v$, $w_m = w$ and $w'_0 = w'$, $w'_p = v$ (in the case of $\lambda(A - v) = 0$ we have $m = 1$ and $p = 1$). By Lemma 2, the order of the pole of the function $(zE - (A - v))^{-1}(w_i, w'_{i'})$ at the point $z = \lambda(A - v)$ is equal to q for any $1 \leqslant i \leqslant m$ and $0 \leqslant i' \leqslant p - 1$. This means that the number $Q_{-q}(w_i, w'_{i'})$ is not equal to zero and therefore is positive for such i and i'. Choosing w_1 and w'_{p-1} as w_* and w'_*, respectively, we obtain the claim. The theorem is proved. □

In the case of a symmetric matrix the statement of Theorem 2 has the following form (for the Laplacian matrix of a weighted undirected graph, it was obtained in Ref. [14]).

Corollary 2. *Let A be an irreducible symmetric matrix with nonnegative entries. Then the multiplicity of the point $\lambda(A - v)$ as an eigenvalue of the original matrix A is one less than that of $\lambda(A - v)$ as an eigenvalue of the submatrix $A - v$ itself.*

We remark that the statement of Corollary 2 can be proved by a direct method. Such a proof is based on the fact that a nonnegative vector cannot be orthogonal to a positive vector.

By means of Theorem 2 we reduce the problem of the determination of the spectral properties for the original matrix to an analogous problem for one of its principal submatrices of co-order one. At first sight, we have not obtained a considerable result. But it is not quite so because the Perron root is not an ordinary eigenvalue. In many important cases the spectral properties of the Perron root of a nonnegative matrix are simply determined by means of its singular digraph (for example, see Theorem 8.3 [5]). For details, we refer the reader to a number of papers concerning nonnegative matrices, including [5,15–26]. In particular, Schneider's theorem (Theorem 6.3 [5]) states that there is just one Jordan block for the Perron root of a nonnegative matrix M iff there is a path in the singular digraph of M that goes through all its vertices (in other words, there is a path in $G(M)$ that passes through all its Perron strong components). From this statement and Theorem 2 we obtain the following criterion.

Corollary 3. *Let A be an irreducible nonnegative matrix. Then the point $\lambda(A - v)$ does not belong to the spectrum of the matrix A iff there is a path in $G(A - v)$ that passes through all its Perron strong components.*

In conclusion we would like to remark that in the general case the statement of Theorem 2 is not true for the other eigenvalues of the matrix $A - v$ belonging to the set $|\lambda| = \lambda(A - v)$. Indeed, let C be any irreducible symmetric matrix with nonnegative entries whose graph $G(C)$ is bipartite. In this case the point $-\lambda(C)$ is a simple eigenvalue. Denote by ζ any eigenvector associated with $-\lambda(C)$. Let us adjoin any positive Perron eigenvector ξ of C to the matrix C as a new column and a new row (we can assume that the new diagonal entry is equal to zero). Since the eigenvector

ζ is orthogonal to the eigenvector ξ, the vector whose vth entry is equal to zero and whose remaining part coincides with the vector ζ is an eigenvector of the new (also irreducible) matrix corresponding to the eigenvalue $-\lambda(C)$:

$$\begin{pmatrix} 0 & \xi \\ \xi & C \end{pmatrix} \begin{pmatrix} 0 \\ \zeta \end{pmatrix} = -\lambda(C) \begin{pmatrix} 0 \\ \zeta \end{pmatrix}.$$

Thus, in this case the change of the spectral properties is not typical. Unfortunately, it is a standard situation that a statement which holds for the Perron root is not valid for the other spectral–circle eigenvalues (see [27]).

4. The spectral properties of the Perron root of a principal submatrix of co-order one as an eigenvalue of the submatrix itself

All the results of this section are based on the following simple lemma.

Lemma 3. *The algebraic multiplicity of the Perron root of a nonnegative matrix A as an eigenvalue of $A - v$ either remains the same or decreases by one*:

$$n_A(\lambda(A)) - 1 \leqslant n_{A-v}(\lambda(A)) \leqslant n_A(\lambda(A)).$$

More precisely, the algebraic multiplicity of the eigenvalue $\lambda(A)$ decreases by one if v belongs to the set of vertices of some Perron strong component of the weighted digraph $G(A)$ and remains the same in the opposite case. We omit the proof of this statement here because it is a simple consequence of the fact that the algebraic multiplicity of the Perron root is equal to the number of the Perron strong components.

Remark 3. For the index of the Perron root the inequality

$$[\mathrm{ind}_A(\lambda(A))/2] \leqslant \mathrm{ind}_{A-v}(\lambda(A)) \leqslant \mathrm{ind}_A(\lambda(A))$$

holds. On the other hand, there are no restrictions to the change of the geometric multiplicity of the Perron root in comparison with the case of an arbitrary eigenvalue of a matrix with complex entries. So, $g_A(\lambda(A)) - 1 \leqslant g_{A-v}(\lambda(A)) \leqslant g_A(\lambda(A)) + 1$. In particular, the geometric multiplicity of the Perron root $\lambda(A)$ as an eigenvalue of $A - v$ can be one more than that of $\lambda(A)$ as an eigenvalue of the original matrix A.

Definition 1. We say that $A - v$ has the smallest Perron root if the Perron root of the submatrix $A - v$ is minimal among all the principal submatrices of co-order one of the matrix A:

$$\lambda(A - v) = \min_{w \in V(A)} \lambda(A - w) \equiv \lambda_{\min}(A).$$

As we have already seen, for any $v \in V(A)$, the interval $(\lambda(A-v), \lambda(A))$ does not contain any eigenvalues of the original matrix A. Of course, the same is true for the interval $(\lambda_{\min}(A), \lambda(A))$ (see other bounds for real eigenvalues of a nonnegative matrix in [28–31]). From this and Theorem 2 it follows that there is only one Jordan block for the Perron root of the submatrix $A-v$ when $\lambda(A-v) > \lambda_{\min}(A)$. The following statement sharpens this simple consequence of Theorem 2.

Proposition 1. *Let A be an irreducible nonnegative matrix. Assume that the Perron root of $A-v$ is not smallest*: $\lambda(A-v) > \lambda_{\min}(A)$. *Then the point $\lambda(A-v)$ is a simple eigenvalue of the matrix $A-v$.*

Proof. Assume that $A-w$ has the smallest Perron root. Since the matrices $A-w$ and $A-w-v$ are nonnegative and the second one is a principal submatrix of the first one, we have $\lambda(A-w-v) \leqslant \lambda(A-w)$. On the other hand, by the choice of the indices v and w, the strict inequality $\lambda(A-w) < \lambda(A-v)$ takes place. Therefore $\lambda(A-w-v) < \lambda(A-v)$. This inequality means that the point $\lambda(A-v)$ is not an eigenvalue of the matrix $A-w-v$. Obviously, the matrix $A-w-v$ coincides with the matrix $A-v-w$, which is a principal submatrix of co-order one of $A-v$. Thus, by Lemma 3, the algebraic multiplicity of the Perron root of the matrix $A-v$ can only be equal to one. The proposition is proved. $\square$

Remark 4. An analogous result is true for any symmetric matrix such that the right edge of its spectrum is a simple eigenvalue. But the Taussky unification problem (see [32]) is not really relevant here: both results follow from two simple facts which hold for both nonnegative and symmetric matrices. The first one states that under passing from A to $A-v$ the algebraic multiplicity of the biggest real eigenvalue of A either remains the same or decreases by one. The second one is the monotonicity property: the biggest real eigenvalue of any principal submatrix is not greater than that of the original matrix.

Remark 5. The absence of an index v such that $\lambda(A-v) > \lambda_{\min}(A)$ means that all the principal submatrices of co-order one have the same Perron root. If for at least one of them, it is not a simple eigenvalue (in other words, for some $w \in V(A)$ the digraph $G(A-w)$ has at least two Perron strong components), then the digraph $G(A)$ must be Hamiltonian and its diameter is one less than its order (see Theorem 2 and Proposition 2′ in [33]).

It is not difficult to construct an irreducible nonnegative matrix such that the Perron root of each of its principal submatrices of co-order one is a simple eigenvalue. For instance, any matrix with positive entries has this property. On the other hand, it is clear that any nonnegative matrix M can be a principal submatrix of co-order one of some irreducible nonnegative matrix (it is sufficient to adjoin new positive row and column to the matrix M for that). By Proposition 1, if the algebraic multiplicity of the Perron root of the matrix M is strictly greater than one, then it must have the smallest Perron root as a principal submatrix of co-order one. Thus, there are no restrictions to the spectral properties of the smallest Perron root: any finite set of natural numbers can be the range of the sizes of its Jordan blocks. Theorem 2 allows us to show only the consistency of the spectral properties of any two submatrices with the smallest Perron root of the same irreducible matrix.

Proposition 2. *Let A be an irreducible nonnegative matrix. Assume that both $A - v$ and $A - w$ have the smallest Perron root: $\lambda(A - v) = \lambda(A - w) = \lambda_{\min}(A)$. Let $n_1 \leqslant \cdots \leqslant n_k$ be the sizes of the Jordan blocks for the Perron root of the submatrix $A - v$ and let $n'_1 \leqslant \cdots \leqslant n'_{k'}$ be those for the Perron root of the submatrix $A - w$. Then*

(1) $k = k'$;
(2) $n_i = n'_i$ *for* $i = 1, \ldots, k - 1$;
(3) $n_k - 1 \leqslant n'_k \leqslant n_k + 1$.

Proof. The first and second statements of the proposition are direct consequences of Theorem 2. Indeed, $n_1, \ldots, n_{k-1}$ is the range of the sizes of the Jordan blocks for the eigenvalue $\lambda(A - v)$ of the matrix A. On the other hand, $n'_1, \ldots, n'_{k'-1}$ is that for the eigenvalue $\lambda(A - w)$ of the matrix A. By assumption, $\lambda(A - v) = \lambda(A - w)$. Thus, $k = k'$ and $n_i = n'_i$ for $i = 1, \ldots, k - 1$.

By Lemma 3, the algebraic multiplicity of the Perron root of the matrix $A - v$ is greater by one than or equal to the algebraic multiplicity of the point $\lambda(A - v)$ as an eigenvalue of the matrix $A - v - w$. The same statement is true for the matrices $A - w$ and $A - w - v$. But $A - v - w = A - w - v$ and, by the assumption of the proposition, $\lambda(A - v) = \lambda(A - w)$. Hence

$$n_1 + \cdots + n_k - 1 \leqslant n'_1 + \cdots + n'_{k'} \leqslant n_1 + \cdots + n_k + 1.$$

We have already shown above that $k = k'$ and $n_i = n'_i$ for $i = 1, \ldots, k - 1$. Thus, inequality (3) takes place for the sizes n_k and n'_k of the largest Jordan blocks. The proposition is proved. □

Remark 6. If A is also symmetric and $k \geqslant 2$ for some submatrix $A - v$, then the Perron root of any other principal submatrix of co-order one of A is strictly greater than that of $A - v$ (see Corollary 3 in Appendix to [33]). For an arbitrary irreducible matrix A with nonnegative entries, the condition $k \geqslant 2$ does not imply that only the submatrix $A - v$ has the smallest Perron root. But even in this case there always exists an index w such that $\lambda(A - w) > \lambda_{\min}(A)$ (see Remark 2). So, all the principal submatrices of co-order one of A cannot have the same Perron root if the condition $k \geqslant 2$ holds for at least one of them.

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Appendix A. Jordan basis corresponding to the eigenvalues contained in the annulus $\{\lambda\colon \lambda_{\min}(A) < |\lambda| \leqslant \lambda(A)\}$

In Appendix we describe the generalized eigenvectors corresponding to the eigenvalues in the domain $|\lambda| > \lambda_{\min}(A)$. Here we shall use the following form of identity (1):

$$\det(A - zE) = (A(v, v) - z - (\xi_v, ((A - v) - zE)^{-1}\eta_v)) \\ \times \det((A - v) - zE).$$

Theorem A.1. *Let λ be an eigenvalue of a square matrix A that does not belong to the spectrum of its submatrix $A - v$ (without loss of generality, we can assume that v is the first index of the matrix A). Then the geometric multiplicity of the eigenvalue λ of the matrix A is equal to one. Moreover, the vectors*

$$\xi_m = \begin{pmatrix} -\delta_{m1} \\ ((A - v) - \lambda E)^{-m}\eta_v \end{pmatrix}, \quad m = 1, \dots, q = \operatorname{ind}_A(\lambda),$$

form a Jordan chain of A corresponding to the eigenvalue λ.

Proof. The first statement of the theorem is a simple consequence of Theorem 1.4.9 [34]. Nevertheless, we give its proof here. Suppose the opposite, i.e. the geometric multiplicity of the eigenvalue λ of the matrix A is greater than one. Then we can easily construct an eigenvector of the matrix A corresponding to the eigenvalue λ that has zero vth entry. It is evident that its restriction to the set $V(A - v)$ is an eigenvector of the matrix $A - v$ corresponding to the same eigenvalue. Thus, the point λ belongs to the spectrum of the principal submatrix $A - v$. But this fact contradicts the initial conditions. So the first statement of the theorem is proved.

The second statement of the theorem is directly verified by considering the action of the matrix $A - \lambda E$ on the vectors defined above. Indeed,

$$\begin{pmatrix} A(v, v) - \lambda & \xi_v \\ \eta_v & (A - v) - \lambda E \end{pmatrix} \begin{pmatrix} -1 \\ ((A - v) - \lambda E)^{-1}\eta_v \end{pmatrix} = 0,$$

$$\begin{pmatrix} A(v,v)-\lambda & \xi_v \\ \eta_v & (A-v)-\lambda E \end{pmatrix} \begin{pmatrix} 0 \\ ((A-v)-\lambda E)^{-2}\eta_v \end{pmatrix}$$
$$= \begin{pmatrix} -1 \\ ((A-v)-\lambda E)^{-1}\eta_v \end{pmatrix},$$

and for $3 \leqslant m \leqslant \mathrm{ind}_A(\lambda)$ we have

$$\begin{pmatrix} A(v,v)-\lambda & \xi_v \\ \eta_v & (A-v)-\lambda E \end{pmatrix} \begin{pmatrix} 0 \\ ((A-v)-\lambda E)^{-m}\eta_v \end{pmatrix}$$
$$= \begin{pmatrix} 0 \\ ((A-v)-\lambda E)^{-m+1}\eta_v \end{pmatrix}.$$

Here we must use the equalities

$$\left(\xi_v, ((A-v)-\lambda E)^{-1}\eta_v\right) - A(v,v) + \lambda = 0,$$

$$\left(\xi_v, ((A-v)-\lambda E)^{-2}\eta_v\right) = -1,$$

and

$$\left(\xi_v, ((A-v)-\lambda E)^{-m}\eta_v\right) = 0, \text{ where } 3 \leqslant m \leqslant \mathrm{ind}_A(\lambda).$$

They are all consequences of the fact that the point $z=\lambda$ is a zero of order $\mathrm{ind}_A(\lambda)$ of the function $A(v,v) - z - (\xi_v, ((A-v)-zE)^{-1}\eta_v)$ when the eigenvalue λ of the matrix A does not belong to the spectrum of the submatrix $A-v$. The proof of the theorem is complete. □

Applying the statement of Theorem A.1 to the transpose-matrix A^{T}, which has the form

$$\begin{pmatrix} A(v,v) & \eta_v \\ \xi_v & (A-v)^{\mathrm{T}} \end{pmatrix},$$

we obtain that the vectors

$$\eta_m = \begin{pmatrix} -\delta_{m1} \\ ((A-v)^{\mathrm{T}}-\lambda E)^{-m}\xi_v \end{pmatrix}, \text{ where } m = 1, \ldots, q = \mathrm{ind}_A(\lambda),$$

form a Jordan chain of A^{T} corresponding to the eigenvalue λ. However, in the general case, these vectors do not form the Jordan chain of A^{T} for λ which is dual to the Jordan chain $\xi_1, \ldots, \xi_q$ of A. We recall that a system of vectors ζ_m, $m = 1, \ldots, q$, is dual to a system of vectors ζ'_m, $m = 1, \ldots, q$, iff $(\zeta_m, \zeta'_{m'}) = \delta_{m\,m'}$. The following statement allows us to construct such a system.

Proposition A.1. *Assume that the conditions of Theorem* A.1 *hold. Let* $\eta'_1, \ldots, \eta'_q$ *be the sequence such that* $\eta'_1 = \eta_1$ *and for any* $k = 1, \ldots, q-1$

$$\eta'_{k+1} = \eta_{k+1} - \alpha_1\eta'_k - \cdots - \alpha_m\eta'_{k-m+1} - \cdots - \alpha_k\eta'_1,$$

where

$$\alpha_m = \frac{\big(\xi_v, ((A-v)-\lambda E)^{-q-m-1}\eta_v\big)}{\big(\xi_v, ((A-v)-\lambda E)^{-q-1}\eta_v\big)}.$$

Then the vectors

$$\frac{1}{\delta_{q1} + (\xi_v, ((A-v)-\lambda E)^{-q-1}\eta_v)}\eta'_q, \ldots, \frac{1}{\delta_{q1} + (\xi_v, ((A-v)-\lambda E)^{-q-1}\eta_v)}\eta'_1$$

form the Jordan chain of A^{T} *for* λ *which is dual to the Jordan chain* $\xi_1, \ldots, \xi_q$ *of* A *for* λ.

Proof. The case of $q = 1$ is trivial. So we shall assume that $q \geqslant 2$. The proof will be done by induction on k. Indeed, the vector η_1 is orthogonal to any vector from the set $\xi_1, \ldots, \xi_{q-1}$ because

$$\begin{aligned}(\eta_1, \xi_1) &= 1 + \Big(\big((A-v)^{\mathrm{T}} - \lambda E\big)^{-1}\xi_v, \big((A-v)-\lambda E\big)^{-1}\eta_v\Big)\\ &= 1 + \Big(\xi_v, \big((A-v)-\lambda E\big)^{-2}\eta_v\Big) = 0,\end{aligned}$$

$$\begin{aligned}(\eta_1, \xi_m) &= \Big(\big((A-v)^{\mathrm{T}} - \lambda E\big)^{-1}\xi_v, \big((A-v)-\lambda E\big)^{-m}\eta_v\Big)\\ &= \Big(\xi_v, \big((A-v)-\lambda E\big)^{-m-1}\eta_v\Big) = 0 \quad \text{for } 2 \leqslant m \leqslant q-1.\end{aligned}$$

So, $(\eta_1, \xi_m) = (\eta_1, \xi_q)\delta_{1\,q-m+1}$, where $(\eta_1, \xi_q) = (\xi_v, ((A-v)-\lambda E)^{-q-1}\eta_v) \neq 0$, and therefore we can consider the case of $k = 1$ as the base of the induction.

Assume now that we have already checked that $(A^{\mathrm{T}} - \lambda E)\eta'_p = \eta'_{p-1}$ for $p = 1, \ldots, k$ (here $\eta'_0 = 0$) and $(\eta'_p, \xi_m) = (\eta_1, \xi_q)\delta_{p\,q-m+1}$, where $p = 1, \ldots, k$ and $m = 1, \ldots, q$. Then

$$(A^{\mathrm{T}} - \lambda E)\eta'_{k+1} = \eta_k - \alpha_1\eta'_{k-1} - \cdots - \alpha_{k-1}\eta'_1 = \eta'_k$$

and therefore

$$\begin{aligned}\big(\eta'_{k+1}, \xi_m\big) &= \big(\eta'_{k+1}, (A-\lambda E)\xi_{m+1}\big) = \big((A^{\mathrm{T}} - \lambda E)\eta'_{k+1}, \xi_{m+1}\big)\\ &= \big(\eta'_k, \xi_{m+1}\big) = \big(\eta_1, \xi_q\big)\delta_{k\,q-m}\end{aligned}$$

for $m \leqslant q-1$.

Moreover,

$$\begin{aligned}&\big(\eta'_{k+1}, \xi_q\big) = \big(\eta_{k+1} - \alpha_1\eta'_k - \cdots - \alpha_k\eta'_1, \xi_q\big) = \big(\eta_{k+1}, \xi_q\big) - \alpha_k\big(\eta_1, \xi_q\big)\\ &\quad = \big(\xi_v, ((A-v)-\lambda E)^{-q-k-1}\eta_v\big) - \alpha_k\big(\xi_v, ((A-v)-\lambda E)^{-q-1}\eta_v\big)\\ &\quad = 0.\end{aligned}$$

So $(\eta'_{k+1}, \xi_m) = (\eta_1, \xi_q)\delta_{k+1\,q-m+1}$ and therefore the step of the induction is done. The proposition is proved. $\square$

Remark A.1. The vector-function $((A-v)-zE)^{-1}\eta_v$ is analytic at the point $z=\lambda$ if and only if the vector η_v belongs to the direct sum of all the generalized eigenspaces of the matrix $A-v$ corresponding to its eigenvalues other than λ (for any η from the direct sum of these generalized eigenspaces, the vector $((A-v)-\lambda E)^{-1}\eta$ is defined as a unique element ζ of this direct sum such that $((A-v)-\lambda E)\zeta=\eta$). An analogous statement is true for the vector-function $((A-v)^{\mathrm{T}}-zE)^{-1}\xi_v$: it is analytic at the point $z=\lambda$ iff the vector ξ_v belongs to the direct sum of all the generalized eigenspaces of the matrix $(A-v)^{\mathrm{T}}$ corresponding to its eigenvalues other than λ. If both of these conditions concerning η_v and ξ_v hold, then $\xi_1,\ldots,\xi_q$ is a Jordan chain of length $q=n_A(\lambda)-n_{A-v}(\lambda)$ for the matrix A and $\eta_1,\ldots,\eta_q$ is that for the matrix A^{T} (both these Jordan chains correspond to the eigenvalue λ). Moreover, the vectors $\eta'_q,\ldots,\eta'_1$ divided by $\delta_{q1}+(\xi_v,((A-v)-\lambda E)^{-q-1}\eta_v)$ form the Jordan chain of A^{T} for λ which is dual to the Jordan chain $\xi_1,\ldots,\xi_q$ of A for λ.

Denote by $L^+(A;w,v)$ the set of walks from w to v in the digraph $G(A)$ all of whose intermediate vertices are different from v. So, for any $\gamma=\{w_i\}_{i=0}^k$ from $L^+(A;w,v)$, we have $w_0=w$, $w_k=v$, and $w_i\neq v$ for $i=1,\ldots,k-1$. Set

$$\xi_v(w;z)=\sum_{\gamma\in L^+(A;w,v)} Q_A(\gamma)z^{\ell(\gamma)}.$$

Since A is a finite matrix, the generating function $\xi_v(w;z)$ has nonzero radius of convergence and therefore defines an analytic function in some neighborhood of zero. Denote by $K(A-v;w,w')$ the set of walks from w to w' in the digraph $G(A-v)$. It is not difficult to verify (see the proof of Lemma 2 in Section 3) that

$$(E-z(A-v))^{-1}(w,w')=\delta_{ww'}+\sum_{\gamma\in K(A-v;w,w')} Q_A(\gamma)z^{\ell(\gamma)}.$$

So we have

$$\begin{aligned}\xi_v(w;z)&=\sum_{\gamma\in L^+(A;w,v)} Q_A(\gamma)z^{\ell(\gamma)}\\&=A(w,v)z+\sum_{w'\neq v}A(w',v)z\sum_{\gamma\in K(A-v;w,w')} Q_A(\gamma)z^{\ell(\gamma)}\\&=z\sum_{w'\neq v}(E-z(A-v))^{-1}(w,w')A(w',v)\\&=z(e_w,(E-z(A-v))^{-1}\eta_v),\end{aligned}$$

where e_w is the vector whose wth entry is equal to one and all the others are equal to zero. This means that $\xi_v(w;z)$ really defines a rational function in the whole complex plane $\mathbb{C}$. Any pole of $\xi_v(w;z)$ coincides with the reciprocal of some eigenvalue of the matrix $A-v$. In particular, for any w the rational function $\xi_v(w;z)$ is analytic in some neighborhood of z if the point z^{-1} does not belong to the spectrum of the matrix $A - v$.

Let $\xi_v(z)$ be the vector whose w-entry is equal to $\xi_v(w; z)$ for $w \neq v$ and whose vth entry is 1. So,

$$\begin{pmatrix} -1 \\ ((A - v) - zE)^{-1}\eta_v \end{pmatrix} = -\xi_v(z^{-1}).$$

Since

$$\frac{\partial}{\partial z}\begin{pmatrix} -1 \\ ((A - v) - zE)^{-1}\eta_v \end{pmatrix} = \begin{pmatrix} 0 \\ ((A - v) - zE)^{-2}\eta_v \end{pmatrix}$$

and

$$\frac{\partial}{\partial z}\begin{pmatrix} 0 \\ ((A - v) - zE)^{-m}\eta_v \end{pmatrix} = m\begin{pmatrix} 0 \\ ((A - v) - zE)^{-m-1}\eta_v \end{pmatrix}$$

for $m \geqslant 1$ we also have

$$\begin{pmatrix} 0 \\ ((A - v) - zE)^{-m-1}\eta_v \end{pmatrix} = -\frac{1}{m!}\frac{\partial^m}{\partial z^m}\xi_v(z^{-1}).$$

Denote by $L^-(A; v, w)$ the set of walks from v to w in the digraph $G(A)$ each of whose intermediate vertices is different from v. So, if $\gamma = \{w_i\}_{i=0}^k$ belongs to $L^-(A; v, w)$, then $w_0 = v$, $w_k = w$ and $w_i \neq v$ for $i = 1, \ldots, k - 1$. Set

$$\eta_v(w; z) = \sum_{\gamma \in L^-(A; v, w)} Q_A(\gamma) z^{\ell(\gamma)}.$$

For any $\gamma = \{w_i\}_{i=0}^k$, denote by γ^{-1} the sequence $\{w_i'\}_{i=0}^k$, where $w_i' = w_{k-i}$. Since $L^-(A; v, w) = L^+(A^{\mathrm{T}}; w, v)^{-1}$ and $Q_A(\gamma) = Q_{A^{\mathrm{T}}}(\gamma^{-1})$, the following representation

$$\eta_v(w; z) = z\big(e_w, \big(E - z(A - v)^{\mathrm{T}}\big)^{-1}\xi_v\big)$$

holds.

Let $\eta_v(z)$ be the vector whose w-entry is equal to $\eta_v(w; z)$ for $w \neq v$ and whose vth entry is 1.

Then we have

$$\begin{pmatrix} -1 \\ ((A - v)^{\mathrm{T}} - zE)^{-1}\xi_v \end{pmatrix} = -\eta_v(z^{-1})$$

and

$$\begin{pmatrix} 0 \\ ((A - v)^{\mathrm{T}} - zE)^{-m-1}\xi_v \end{pmatrix} = -\frac{1}{m!}\frac{\partial^m}{\partial z^m}\eta_v(z^{-1}), \text{ where } m \geqslant 1.$$

By definition, the set $L^-(A; v, v) = L^+(A; v, v)$ consists of all the v-closed walks which do not have the vertex v as an intermediate vertex. Denote by $L(A; v)$ this set. Let

$$\phi_{A,v}(z) = \sum_{\gamma \in L(A;v)} Q_A(\gamma) z^{\ell(\gamma)-1}.$$

It is not difficult to check that $\phi_{A,v}(z) = A(v, v) + z(\xi_v, (E - z(A - v))^{-1}\eta_v)$ and therefore $(\xi_v, ((A - v) - zE)^{-1}\eta_v) = A(v, v) - \phi_{A,v}(z^{-1})$. So for $m \geqslant 1$ we have

$$\left(\xi_v, ((A - v) - zE)^{-m-1}\eta_v\right) = -\frac{1}{m!}\frac{\partial^m}{\partial z^m}\phi_{A,v}(z^{-1}).$$

Suppose now that the matrix A is irreducible and all its entries are nonnegative. Let v be an index such that $\lambda(A - v) = \lambda_{\min}(A)$. Then for any w the radii of convergence of the generating functions $\xi_v(w; z)$ and $\eta_v(w; z)$ are not less than $\lambda_{\min}(A)^{-1}$. Moreover, the radius of convergence of the generating function $\phi_{A,v}(z)$ is equal to $\lambda_{\min}(A)^{-1}$. Therefore, we can use these generating functions for determining the generalized eigenvectors corresponding to any eigenvalue from the annulus $\{\lambda\colon \lambda_{\min}(A) < |\lambda| \leqslant \lambda(A)\}$.

Corollary A.1. *Let A be an irreducible nonnegative matrix. Then any eigenvalue λ contained in the annulus $\{\lambda\colon \lambda_{\min}(A) < |\lambda| \leqslant \lambda(A)\}$ has geometric multiplicity one, and for any index v such that $\lambda(A - v) = \lambda_{\min}(A)$ the vectors*

$$\xi_m = -\frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial z^{m-1}}\xi_v(z^{-1})\Big|_{z=\lambda}, \text{ where } m = 1, \ldots, \operatorname{ind}_A(\lambda),$$

and

$$\eta_m = -\frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial z^{m-1}}\eta_v(z^{-1})\Big|_{z=\lambda}, \text{ where } m = 1, \ldots, \operatorname{ind}_A(\lambda),$$

form Jordan chains of the matrices A and A^{T}, respectively, corresponding to the eigenvalue λ. Let $\eta'_1, \ldots, \eta'_q$ be the sequence such that

$$\eta'_1 = \eta_1 \text{ and } \eta'_{k+1} = \eta_{k+1} - \alpha_1\eta'_k - \cdots - \alpha_m\eta'_{k-m+1} - \cdots - \alpha_k\eta'_1,$$

where

$$\alpha_m = \frac{q!}{(q+m)!}\frac{\frac{\partial^{q+m}}{\partial z^{q+m}}\phi_{A,v}(z^{-1})\Big|_{z=\lambda}}{\frac{\partial^q}{\partial z^q}\phi_{A,v}(z^{-1})\Big|_{z=\lambda}}.$$

Then

$$\frac{q!}{\delta_{q1} - \frac{\partial^q}{\partial z^q}\phi_{A,v}(z^{-1})|_{z=\lambda}}\eta'_q, \ldots, \frac{q!}{\delta_{q1} - \frac{\partial^q}{\partial z^q}\phi_{A,v}(z^{-1})|_{z=\lambda}}\eta'_1$$

is the Jordan chain of A^{T} for λ which is dual to the Jordan chain $\xi_1, \ldots, \xi_q$ of A for λ.

Remark A.2. For any two indices such that the Perron roots of the corresponding principal submatrices of co-order one are equal to $\lambda_{\min}(A)$, the application of Corollary A.1 can give different Jordan chains for the eigenvalue λ. So the systems of vectors $\xi_1, \ldots, \xi_q$ and $\eta_1, \ldots, \eta_q$ need not be the same for all such indices.

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